Appl. Math. Lett. Vol. 1, No. 3, pp. 299-302, 1988
Printed in Great Britain
0893-9659/88 $3.00 + 0.00
Pergamon Press plc

Numerical Experiments on a Domain Decomposition Algorithm for Nonlinear Elliptic Boundary Value Problems

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Domain Decomposition

In this note we present numerical experiments on a domain decomposition algorithm for nonlinear elliptic boundary value problems in planar domains. There has recently been much progress in the development of such algorithms for linear elliptic problems ([1],[2],[3],[4]). These have focused on variational characterizations of the problem and on the preconditioning of the Schur complement associated with the decomposition. Although these could be used as part of a global Newton-type iterative scheme to solve a nonlinear problem, we choose the alternate path of first decomposing the problem and then applying an iterative method. Our motivation for this is two-fold; first, we expect it will lead to algorithms which will require less communication between subproblems, an attractive property for implementations on parallel processors; second, this approach has even been found to be more efficient for serial computations in some cases [6]. The essential step in this method is the solution of what we call the basic equations, a nonlinear analogue of the Schur complement problem. We are particularly concerned with the choice of boundary conditions at boundaries where subdomains intersect and their effect on the basic equations. Note that the methods we propose may also be applied to linear problems and seem to be novel in that context.

We restrict ourselves here to the study of a decomposition into two domains:

$$\Omega = \Omega_1 \cup \Omega_2, \tag{1}$$

$$L(u_i) = f, \quad x \in \Omega_i, \quad A_i \frac{\partial u_i}{\partial n} + B_i u_i = r_i, \quad x \in \partial\Omega_i. \tag{2}$$

For definiteness we assume that L is a nonlinear differential operator of second order. Parts of the boundary will be artificial boundaries, where subdomains, Ω_i, intersect. For these the form of (A_i, B_i) can be chosen freely and, in general, they may be operators in an appropriate space of functions on the boundary. Indeed, we advocate the use of *nonlocal conditions*. In order that the subdomain solutions define a solution throughout, certain continuity conditions must be satisfied:

$$(A_1 \frac{\partial}{\partial n} + B_1) u_2(r_2) = r_1, \quad (A_2 \frac{\partial}{\partial n} + B_2) u_1(r_1) = r_2, \tag{3}$$

where $x \in \partial\Omega_1 \cap \partial\Omega_2$.

We now introduce discrete approximations of problem (2). Let v_i be the approximation to u_i and w_i be the approximation to r_i. We then have:

$$F_1(v_1, w_1) = 0, \quad F_2(v_2, w_2) = 0. \tag{4}$$

The continuity equations are now replaced by:

$$G_1(v_2) = w_1, \quad G_2(v_1) = w_2. \tag{5}$$

Following the ideas presented in [6], we call the w_i's *basic* variables and the v_i's *nonbasic* variables. Correspondingly, equations (4) are nonbasic and (5) are basic. In some neighborhood of an isolated solution of the entire system we expect that the nonbasic equations implicitly define the nonbasic variables as functions of the basic variables:

$$v_i = T_i(w_i). \tag{6}$$

Of course, T_i is evaluated in practice by solving the nonbasic equations. As they are uncoupled, this may be carried out on independent processors. To complete the solution process, we must solve the basic equations for the basic variables:

$$G_1(T_2(w_2)) = w_1, \quad G_2(T_1(w_1)) = w_2. \tag{7}$$

Convergence results for a variety of Newton-type iterative methods for the solution of the basic equations are given in [6]. In the experiments we have used a quasi-Newton algorithm based on Broyden's update. That is:

$$\begin{pmatrix} I & J_{12}^{(n)} \\ J_{21}^{(n)} & I \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} = \begin{pmatrix} G_1(T_2(w_2^{(n)})) - w_1^{(n)} \\ G_2(T_1(w_1^{(n)})) - w_2^{(n)} \end{pmatrix}, \tag{8}$$

$$w_i^{(n+1)} = w_i^{(n)} + dw_i, \tag{9}$$

$$J_{ij}^{(n+1)} = J_{ij}^{(n)} + (w_i^{(n+1)} - G_i(T_j(w_j^{(n+1)}))) \frac{dw_j^T}{dw_j^T dw_j}. \tag{10}$$

In the future, experiments involving other methods, especially of conjugate gradient type, will be considered.

Choice of Boundary Conditions

The choice of boundary operators, A_i and B_i, directly determines the properties of the Jacobian of the basic equations. In particular, we see that the matrix J_{ij} is a discrete approximation of:

$$S_{ij} = -(A_i \frac{\partial}{\partial n} + B_i) \frac{\partial u_j}{\partial r_j}, \quad x \in \partial\Omega_1 \cap \partial\Omega_2. \tag{11}$$

Clearly, the conditions should be chosen to minimize the norm of these operators. We take A_i to be the identity operator and insist that B_i be no stronger than a tangential derivative. We then expect, using standard trace theorems, that S_{ij} is a mapping from $H_{s-\frac{3}{2}}(\partial\Omega_1 \cap \partial\Omega_2)$ to itself when u is in the Sobolev space $H_s(\Omega)$. For rectangular domains and separable linear operators, optimal choices of the boundary conditions can be made. Here our analysis is similar in spirit with that presented in [5]. For example, consider Poisson's equation in the infinite channel, $(-\infty, \infty) \times (y_0, y_1)$, with homogeneous Dirichlet conditions on the walls and take $x = 0$ as the subdomain boundaries. Then the choice,

$$B_1 = \sqrt{-\frac{\partial^2}{\partial y^2}}, \quad B_2 = -B_1, \tag{12}$$

is optimal. That is, $S_{ij} = 0$. The natural extension of this idea to the problem under consideration is to approximate the principal part of the differential operator by a separable operator and use the optimal conditions which it determines. Precise estimates and convergence results for this procedure will be published elsewhere. An area for future research is the incorporation of effects due to nonlinearities and variable coefficients. We note that the use of a bounded, local operator B_i is likely to be unreliable as the mapping S_{ij} can then be expected to have eigenvalues near one.

Numerical Results

We now present the results of some numerical experiments, mainly using the boundary conditions in (12). The particular equation we have solved is:

$$\nabla^2 u + f(u, x, \lambda) = 0, \ (x, y) \in (0., 1.) \times (0, .5), \tag{13}$$

$$u = 0, \quad x = 0, 1 \text{ or } y = 0, .5. \tag{14}$$

The subdomain boundary was taken at $x = .5$ and f was given by:
• Problem A: $f = \lambda e^u$. • Problem B: $f = 2x\lambda e^u$. • Problem C: $f = 2(x - .5)\lambda e^u$.

The following is a tabulation of results obtained using a microVAX II and a grid of 11 by 20. Second order centered finite differences were employed and the initial guesses for all quantities were zero.

Problem	λ	Inner Iterations	Outer Iterations	Time
A	2.	11	4	41.2
A	8.	14	5	56.5
A	14.	16	5	70.6
B	2.	11	4	43.6
B	8.	14	5	59.7
B	14.	16	5	74.7
C	10.	13	5	52.3
C	34.	15	5	67.6
C	58.	21	6	105.3

Convergence for Decomposition into Two Domains

We note that experiments were carried out with local operators also. Often there was no convergence and when the iterations did converge it was typically very slow. For Problem A we halved the mesh widths and found no degradation in the convergence rates.

We also considered a problem with an advective term:

$$\nabla^2 u + \gamma \frac{\partial u}{\partial x} + 10e^u = 0. \tag{15}$$

In this case the use of (12) was not particularly successful for large values of γ. A better approach is to employ the optimal condition corresponding to the linear operator:

$$\nabla^2 + \gamma \frac{\partial}{\partial x}. \tag{16}$$

Then B_i is symmetric with the same eigenfunctions as $\frac{\partial^2}{\partial y^2}$ but with eigenvalues, ω_i, given by:

$$\omega_1 = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 + 4d^2}\right), \quad \omega_2 = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2 + 4d^2}\right). \tag{17}$$

Here, d^2 is an eigenvalue of $-\frac{\partial^2}{\partial y^2}$. Tabulated below are the results.

Boundary Condition	γ	Inner Iterations	Outer Iterations	Time
12	10.	28	11	127.5
12	25.	93	41	408.3
17	10.	14	5	63.3
17	25.	14	5	63.2
17	50.	12	4	54.7
17	100.	12	4	53.9
17	250.	14	5	60.5
17	500.	20	9	76.1

A Problem with Advection

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